EVOLUTION EFFECTS ON THE NUCLEON DISTRIBUTION AMPLITUDE

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Abstract

We study the Brodsky-Lepage evolution equation for the nucleon and construct an eigenfunction basis by including contributions of up to polynomial order 9. By exployting the permutation symmetry P_{13} of these eigenfunctions, a basis of symmetrized Appell polynomials can be constructed in which the diagonalization of the evolution kernel is considerably simplified. The anomalous dimensions are calculated and found to follow a power-law behavior. As an application, we consider the Brodsky-Huang-Lepage ansatz. An algorithm is developed to properly incorporate such higher order contributions in a systematic way.

I. GENERAL FRAMEWORK

The momentum scale dependence of the nucleon distribution amplitude is given by the Brodsky-Lepage evolution equation[1]

$$x_1 x_2 x_3 \left[\frac{\partial}{\partial \xi} \tilde{\Phi}_i(x_i, \xi) + \frac{3}{2} \frac{C_F}{\beta} \tilde{\Phi}(x_i, \xi) \right] = \frac{C_B}{\beta} \int_0^1 [dy] V[x_i, y_i] \tilde{\Phi}(y_i, \xi), \tag{1}$$

with $\xi = \ln \ln \frac{Q^2}{\Lambda^2_{QCD}}$, $C_B = \frac{N_c+1}{2N_c} = \frac{2}{3}$ and $C_F = \frac{N_c^2-1}{2N_c} = \frac{4}{3}$ the bosonic and fermionic Casimir-operators of $SU(N)_{color}$, respectively; $\beta = 11 - \frac{2}{3} N_F = 9$ being the Gell-Mann and Low function. The asymptotic solution of this equation is $\Phi_{AS} = 120\, x_1 x_2 x_3$. To leading order in α_s, the interaction kernel between quark pairs $\{i, j\}$ is given in Ref. 1. Factorization of Eq. (1) leads to

$$-\eta\, x_1 x_2 x_3\, \tilde{\Phi}(x_i, \xi) = \int_0^1 [dy] V[x_i, y_i] \tilde{\Phi}(y_i, \xi) \quad \text{and} \quad x_1 x_2 x_3 \left[\frac{\partial}{\partial \xi} \tilde{\Phi}(x_i, \xi) + \gamma \tilde{\Phi}(x_i, \xi) \right] = 0 \tag{2}$$

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with $\gamma = \frac{3}{2}\frac{C_F}{\beta} + \eta\frac{C_B}{\beta}$ and the integration measure is defined by $\int_0^1 [dx] \equiv \int_0^1 dx_1 \int_0^{1-x_1} dx_2$ $\int_0^1 dx_3\ \delta(1-x_1-x_2-x_3)$. [In the following we use the operator $\int_0^1 [dy]\, V[x_i, y_i] \mapsto \hat{V}$, which commutes with the permutation operator $\hat{P}_{13}$ $(x_1 \Leftrightarrow x_3)$.] The evolution behavior of the eigenfunctions comes from the sensitivity on the transverse momentum integration[1] arising from the gluon exchange kernels. It can be expressed in the form $\tilde{\Phi}(x_i, Q^2) = \ln^{-\gamma_F}\left(\frac{Q^2}{\Lambda^2_{QCD}}\right) \tilde{\Phi}(x_i)$. The representation of the evolution kernel is conveniently described in terms of Appell polynomials, $\tilde{\mathcal{F}}_{mn}(x_1, x_3)$, which constitute an orthogonal basis with weight[1] $\Phi_{AS}(x_i)/120$. It was shown in Ref. 2 that in this basis, $\hat{V}$ becomes block-diagonal with respect to different polynomial orders $m+n$. Because of $[\hat{P}_{13}, \hat{V}] = 0$, it is useful[3,4] to define a symmetrized basis of Appell polynomials

$$\tilde{\mathcal{F}}_{mn}(x_1, x_3) = (1/2)\ (\mathcal{F}_{mn}(x_1, x_3) \pm \mathcal{F}_{nm}(x_1, x_3)) \quad \text{for } (m \geq n\ /\ m < n). \tag{3}$$

The particular importance of this basis lies in the fact that $\hat{V}$ is block-diagonal within a particular order for different symmetry classes $(S_n = \pm 1)$ with respect to $\hat{P}_{13}$. As a result, it is possible to analytically diagonalize $\hat{V}$ up to order 7 (cf. Tab. I) [up to order $(\mathcal{O}(n) =)\, M = 9$ this is done in Refs. 3,4]. The eigenvalues of any order $\mathcal{O}(n)$ and for a specific symmetry class S_n form a "multiplet"-like set (see Fig. 1), which follows an approximate power-law: $\gamma_n = 0.37\, \mathcal{O}(n)^{0.565}$ that differs from that of scalar (S=0) and vector (S=1) mesons[3,4]. This observation is in contrast to the claims of Ref. 5. The eigenfunctions of the evolution equation have properties of a commutative group

$$\tilde{\Phi}_k(x_i)\, \tilde{\Phi}_n(x_i) = \sum_{l=0}^{\infty} F^l_{kn} \tilde{\Phi}_l(x_i) \quad \text{with} \quad |\mathcal{O}(k) - \mathcal{O}(n)| \leq \mathcal{O}(l) \leq \mathcal{O}(k) + \mathcal{O}(n) \tag{4}$$

and obey the above triangle relation. The structure coefficients[3,4] F^l_{kn} of the group are calculated by $F^l_{kn} = N_l \int_0^1 [dx]\, x_1 x_3 (1-x_1-x_3)\, \tilde{\Phi}_k(x_i) \tilde{\Phi}_n(x_i) \tilde{\Phi}_l(x_i)$, with the particularly important case $F^0_{kk} = \frac{N_0}{N_k}$.

II. APPLICATIONS

The evolution effect of the nucleon distribution amplitude is important for the calculation of various form factors at intermediate values of the momentum transfer $Q^2 \approx 10 \div 30\, GeV^2/c^2$ (see Ref. 1,3–5). In this range, $\alpha_S(Q^2)$ tends to diverge whereas the Q^2 evolution of the distribution amplitude, given by

$$\Phi_N(x_i, Q^2) = \Phi_{AS}(x_i) \left(\sum_{n=0}^{n_{max}} B_n(Q^2) \tilde{\Phi}_n(x_i) \right), \tag{5}$$

significantly reduces the value of the form factors by more than $\approx 30\%$. In the above representation of Φ_N, contributions of eigenfunctions up to polynomial order 3 have been studied[5,6]. By incorporating on the rhs of eq. (5) (c.f. Ref. 7) the factor $f(x_i, \lambda_j) = e^{-\lambda_1^2 \left(\sum_{i=1}^{3} \frac{1}{x_i} - \lambda_2^2\right)}$, eigenfunctions of higher orders can be taken into account having recourse to a Brodsky-Huang-Lepage type of ansatz. This extended ansatz for $\lambda_1 =$

0.03 and $\lambda_2 = 3$ has been studied in Ref. 5. In order to determine its evolution behavior, one has to project[7] Φ_N on the eigenfunctions $\tilde{\Phi}_n$. For this purpose, f can be expanded in the basis of eigenfunctions of the nucleon evolution equation

$$f(x_i, \lambda_j) = \sum_k c_k(\lambda_j)\, \tilde{\Phi}_k(x_i), \quad \text{with} \quad c_k(\lambda_j) = N_k \int_0^1 [dx]\, x_1 x_2 x_3\, f(x_i, \lambda_j) \tilde{\Phi}_k(x_i). \tag{6}$$

This expansion and the properties of products of eigenfunctions given by the structure coefficients F^l_{nk} lead to

$$\Phi_N(x_i) = \Phi_{AS}(x_i) \left(\sum_l \hat{B}_l\, \tilde{\Phi}_l(x_i) \right) \quad \text{with} \quad \hat{B}_l = \left(\sum_{n,k} B_n c_k F^l_{nk} \right). \tag{7}$$

In the last equation the evolution of the distribution amplitude is fully determined by the scale-dependence of the expansion coefficients $\hat{B}_l(Q^2) = \hat{B}_l(\mu^2) \left(\frac{\alpha_S(Q^2)}{\alpha_S(\mu^2)}\right)^{\gamma_l}$, which are a complicated mixture of the original expansion coefficients B_n and the projection coefficients c_k of the extended ansatz. In contrast to the assumption[5]

$$\Phi_N(x_i, Q^2) = \Phi_{AS}(x_i) \left(\sum_{n=0}^{n_{max}} B_n(\mu^2) \left(\frac{\alpha_S(Q^2)}{\alpha_S(\mu^2)} \right)^{\gamma_n} \tilde{\Phi}_n(x_i) \right) f(x_i, \lambda_j), \tag{8}$$

the evolution of the distribution amplitude is *not* found to be determined by the scale-dependence $B_n(Q^2) = B_n(\mu^2) \left(\frac{\alpha_S(Q^2)}{\alpha_S(\mu^2)}\right)^{\gamma_n}$. The deviation of the approximation[5] from the correct evolution behavior can be expressed by the Q-dependent ratio

$$\mathcal{M}_l(Q^2) = \left(\hat{B}_l(\mu^2) \left(\frac{\alpha_S(Q^2)}{\alpha_S(\mu^2)} \right)^{\gamma_l} \right) \Bigg/ \left(\sum_{n,k} B_n(\mu^2) \left(\frac{\alpha_S(Q^2)}{\alpha_S(\mu^2)} \right)^{\gamma_n} c_k F^l_{nk} \right). \tag{9}$$

TABLE I. Orthogonal eigenfunctions $\tilde{\Phi}_n(x_1, x_2, x_3) = \sum_{lk} a^n_{kl}\, x_1^k x_3^l$ of the nucleon evolution equation (represented by the coefficient matrix a^n_{kl} with $a^n_{kl} = S_n\, a^n_{lk}$; $a^n_{22} = 0$ for all n). The normalization is given by $\int_0^1 [dx]\, x_1 x_2 x_3\, \tilde{\Phi}_k(x_i) \tilde{\Phi}_n(x_i) = (N_n)^{-1}\, \delta_{kn}$.

n	M	S_n	γ_n	η_n	N_n
0	0	1	$\frac{2}{27}$	-1	120
1	1	-1	$\frac{26}{81}$	$\frac{2}{3}$	1260
2	1	1	$\frac{10}{27}$	1	420
3	2	1	$\frac{38}{81}$	$\frac{5}{3}$	756
4	2	-1	$\frac{46}{81}$	$\frac{7}{3}$	34020
5	2	1	$\frac{16}{27}$	$\frac{5}{2}$	1944
6	3	1	$\frac{115-\sqrt{97}}{162}$	$\frac{-(-79+\sqrt{97})}{24}$	$\frac{4620\,(485+11\,\sqrt{97})}{97}$
7	3	1	$\frac{115+\sqrt{97}}{162}$	$\frac{79+\sqrt{97}}{24}$	$\frac{4620\,(485-11\,\sqrt{97})}{97}$
8	3	-1	$\frac{559-\sqrt{4801}}{810}$	$\frac{-(-379+\sqrt{4801})}{120}$	$\frac{27720\,(33607-247\,\sqrt{4801})}{4801}$
9	3	-1	$\frac{559+\sqrt{4801}}{810}$	$\frac{379+\sqrt{4801}}{120}$	$\frac{27720\,(33607+247\,\sqrt{4801})}{4801}$
10	4	-1	$\frac{346-\sqrt{1081}}{405}$	$\frac{-(-256+\sqrt{1081})}{60}$	$\frac{196560\,(7567-13\,\sqrt{1081})}{1081}$
11	4	-1	$\frac{346+\sqrt{1081}}{405}$	$\frac{256+\sqrt{1081}}{60}$	$\frac{196560\,(7567+13\,\sqrt{1081})}{1081}$

n	a^n_{00}	a^n_{10}	a^n_{20}	a^n_{11}	a^n_{30}	a^n_{21}	a^n_{40}	a^n_{31}
0	1	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0
2	-2	3	0	0	0	0	0	0

3	2	-7	8	4	0	0	0	0
4	0	1	$-\frac{4}{3}$	0	0	0	0	0
5	2	-7	$\frac{14}{3}$	14	0	0	0	0
6	1	-6	$\frac{41+\sqrt{97}}{4}$	$\frac{3\left(31-\sqrt{97}\right)}{4}$	$\frac{-5\left(17+\sqrt{97}\right)}{16}$	$\frac{-5\left(31-\sqrt{97}\right)}{8}$	0	0
7	1	-6	$\frac{41-\sqrt{97}}{4}$	$\frac{3\left(31+\sqrt{97}\right)}{4}$	$\frac{-5\left(17-\sqrt{97}\right)}{16}$	$\frac{-5\left(31+\sqrt{97}\right)}{8}$	0	0
8	0	1	-3	0	$\frac{601+\sqrt{4801}}{264}$	$\frac{59-\sqrt{4801}}{44}$	0	0
9	0	1	-3	0	$\frac{601-\sqrt{4801}}{264}$	$\frac{59+\sqrt{4801}}{44}$	0	0
10	0	1	-5	0	$\frac{379+\sqrt{1081}}{48}$	$\frac{61-\sqrt{1081}}{8}$	$\frac{-\left(159+\sqrt{1081}\right)}{40}$	$\frac{-\left(61-\sqrt{1081}\right)}{8}$
11	0	1	-5	0	$\frac{379-\sqrt{1081}}{48}$	$\frac{61+\sqrt{1081}}{8}$	$\frac{-\left(159-\sqrt{1081}\right)}{40}$	$\frac{-\left(61+\sqrt{1081}\right)}{8}$

This deviation at intermediate Q^2 is found to be of order 0.9 for $\mathcal{O}(l) \leq 3$ and increases exponentially for higher orders. Since in the ansatz of Ref. 5 $\gamma_0 \geq \gamma_n \geq \gamma_9$, one can approximate

$$\mathcal{M}_l(Q^2) \mapsto \tilde{\mathcal{M}}_l(Q^2) = \left(\hat{B}_l(\mu^2)\left(\frac{\alpha_S(Q^2)}{\alpha_S(\mu^2)}\right)^{\gamma_l}\right) \Big/ \left(\left(\frac{\alpha_S(Q^2)}{\alpha_S(\mu^2)}\right)^{\bar{\gamma}} \sum_{n,k} B_n(\mu^2) c_k F_{nk}^l\right). \quad (10)$$

FIG. 1. The eigenvalues of the evolution equation (for $S_n = 1$) vs. the corresponding order $\mathcal{O}(n) = M$ (solid line).

FIG. 2. Deviation measure $\tilde{\mathcal{M}}_l(Q^2 = 30\ GeV^2/c^2)$ for the two extreme cases $\bar{\gamma} = \gamma_0$ and $\bar{\gamma} = \gamma_9$.

Using the power-law behavior for γ_l, Fig. 2 shows the exponetially increasing deviation of the approximation of Ref. 5 compared to the Q^2-scaling behavior based on the renormalization group equation. This comparison shows that it is important to project the nucleon distribution amplitude Φ_N, as in the case of mesons[7], on the eigenfunctions of the evolution equation. For this purpose, powerful tools have been developed and the basis of eigenfunctions $\tilde{\Phi}_n$ has been extended up to polynomial order 9. Using analytical and numerical algorithms developed in Refs. 3,4 the calculation of higher order eigenfunctions up to any desired precision is shown to be possible.

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